18.152 Midterm exam
due April 6th 9:30 am

1. Preliminary

Given a set $A \subset \mathbb{R}^n$ and a function $u : A \to \mathbb{R}$, we say that

(a) $u \in C(A)$ if u is continuous in A,
(b) $u \in D^k(A)$ if u is k-times differentiable in A,
(c) $u \in C^k(A)$ if u is k-times differentiable and its k-th order derivatives are continuous in A,
(d) $u \in C^\infty(A)$ if u is smooth (∞-many times differentiable) in A,
(e) $u \in C^{0,1}(A)$ if u is locally Lipschitz continuous in A, (Definition 1)
(f) $u \in C^{k,1}(A)$ if u is k-times differentiable and its k-th order derivatives are locally Lipschitz continuous in A.

Definition 1. *We say that $u : A \to \mathbb{R}$ is Lipschitz continuous in A if there exists some constant C_A such that*

$$|u(x) - u(y)| \leq C_A|x - y|, \tag{1}$$

holds for all $x, y \in A$.

We say that $u : A \to \mathbb{R}$ is locally Lipschitz continuous in A if given any compact subset $K \subset A$, u is Lipschitz continuous in K.

We recall a version of the integration by parts.

Theorem 2 (Integration by parts)**.** *A bounded open set $\Omega \subset \mathbb{R}^n$ has the smooth boundary $\partial\Omega$. Then,*

$$\int_\Omega u_i(x)dx = \int_{\partial\Omega} u(\sigma)\nu_i(\sigma)d\sigma, \tag{2}$$

where $\nu_i = \langle \nu, e_i \rangle$.

Proof. We define $V : \Omega \to \mathbb{R}^n$ by $V(x) = u(x)e_i$. Then, the divergence theorem implies

$$\int_\Omega u_i(x)dx = \int_\Omega \text{div}V(x)dx = \int_{\partial\Omega} \langle V(\sigma), \nu(\sigma)\rangle d\sigma = \int_{\partial\Omega} u(\sigma)\nu_i(\sigma)d\sigma. \tag{3}$$

□

2. Laplace equation

Let $\Omega = B_1(0) \subset \mathbb{R}^2$. Given $f \in C^{0,1}(\Omega)$, we define

$$u(x) = -\int_\Omega G(x,y)f(y)dy. \tag{4}$$

Problem 1 (4 points)**.** *Show that*

$$\int_\Omega G(x,y)dy = \frac{1}{2}\left(1-|x|^2\right), \tag{5}$$

holds for $|x| \leq 1$*.*

Problem 2 (4 points)**.** *Show that the following holds in* $\overline{\Omega}$*,*

$$|u(x)| \leq \tfrac{1}{2}\left(1-|x|^2\right)\sup_\Omega |f|. \tag{6}$$

In particular, $u = 0$ *on* $\partial\Omega$*.*

Theorem 3. *u is differentiable in Ω. Moreover, for each $i = 1, 2$, $\frac{\partial}{\partial x_i}u(x) = v_i(x)$ holds in Ω, where $v_i(x)$ is given by*

$$v_i(x) = -\int_\Omega \frac{\partial}{\partial x_i}G(x,y)f(y)dy. \tag{7}$$

Proof. We choose some function $\rho \in C^1(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $0 \leq \rho' \leq 2$, $\rho(t) \leq 1$ for $t \leq 1$ and $\rho(t) = 0$ for $t \geq 2$. Then, given $\epsilon > 0$ we define

$$w_\epsilon(x) = -\int_\Omega \left[\Phi(x-y)\rho_\epsilon + \varphi(x,y)\right] f(y)dy. \tag{8}$$

where $\rho_\epsilon = \rho(|x-y|/\epsilon)$. If $B_{2\epsilon}(x) \subset \Omega$ then $w_\epsilon \in C^1(\Omega)$ and

$$|u - w_\epsilon| \leq \int_{B_{2\epsilon}(x)} \Phi(x-y)|1-\rho_\epsilon||f(y)|dy \leq C\epsilon^2(1+|\log\epsilon|)\sup|f|, \tag{9}$$

and

$$|v_i - \tfrac{\partial}{\partial x_i}w_\epsilon| \leq \int_{B_{2\epsilon}(x)} \left|\tfrac{\partial}{\partial x_i}\Phi(x-y)(1-\rho_\epsilon)\right| |f(y)|dy \tag{10}$$

$$\leq \sup|f| \int_{B_{2\epsilon}(x)} |\tfrac{\partial}{\partial x_i}\Phi(x-y)| + \tfrac{2}{\epsilon}|\Phi(x-y)|dy \tag{11}$$

$$\leq C\epsilon(1+|\log\epsilon|)\sup|f|. \tag{12}$$

Hence, in any compact subset in Ω, w_ϵ and $\frac{\partial}{\partial x_i}$ uniformly converge to u and v_i. Thus u is differentiable and $\frac{\partial}{\partial x_i}u = v_i$. □

Problem 3 (2 point)**.** *Verify $u \in C(\overline{\Omega})$ by using Problem 2 and Theorem 3.*

Given $i, j \in \{1, 2\}$, we define $v_{ij}(x)$ by

$$v_{ij}(x) = -\int_{\Omega} \left(\tfrac{\partial^2}{\partial x_i \partial x_j} G(x,y)\right)[f(y) - f(x)]dy \tag{13}$$

$$+ f(x) \int_{\partial\Omega} \left(\tfrac{\partial}{\partial x_i} G(x,y)\right) \nu_j(\sigma) d\sigma, \tag{14}$$

where $\nu_j(\sigma) = \langle \nu(\sigma), e_j \rangle = \langle \sigma, e_j \rangle = \sigma_j$.

Problem 4 (6 points)**.** *Given $i, j \in \{1, 2\}$, $x \in \Omega$ and $\epsilon > 0$ such that $B_{2\epsilon}(x) \subset \Omega$, we define*

$$w_\epsilon(x) = \int_{\Omega} \left[\varphi(x,y) + \rho_\epsilon(x,y) \tfrac{\partial}{\partial x_i} \Phi(x-y)\right] f(y) \tag{15}$$

where ρ_ϵ is given in the proof of Theorem 3. Then, show that there exists some constant C such that

$$|u_i(x) - w_\epsilon| \leq C\epsilon, \qquad |v_{ij}(x) - \tfrac{\partial}{\partial x_j} w_\epsilon| \leq C\epsilon. \tag{16}$$

Hint 1 : Since $f \in C^{0,1}(\Omega)$, if $B_\delta(x) \subset \Omega$ then there exists some C_0 such that

$$|f(x_1) - f(x_2)| \leq C_0 |x_1 - x_2|, \tag{17}$$

holds for $x_1, x_2 \in B_\delta(x)$.

Hint 2 : You may use Theorem 2.

The result of Problem 4 implies

Theorem 4. *$u \in D^2(\Omega)$ and $\frac{\partial^2}{\partial x_i \partial x_j} u = v_{ij}$ holds in Ω.*

Problem 5 (2 point)**.** *Show that $\Delta u = f$ holds in Ω.*

Problem 6 (2 point)**.** *Show that given $g \in C^{2,1}(\overline{\Omega})$ and $f \in C^{0,1}(\Omega)$, there exists a unique $u \in D^2(\Omega) \cap C(\overline{\Omega})$ satisfying $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$.*

Remark 5. *If $f \in C^{0,1}(\overline{\Omega})$, then we have $u \in C^{2,1}(\overline{\Omega})$. The proof is given in [Gilbarg-Trudinger] section 4.*

3. Liouville theory

Problem 7 (6 point)**.** *Suppose that a positive function* $u \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ *is harmonic. Show that* u *is a constant function.*

Problem 8 (2 point)**.** *Find a non-constant positive harmonic function* $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

Problem 9 (6 point)**.** *Suppose that a harmonic function* $u \in C^\infty(\overline{\mathbb{R}^2_+})$ *satisfies* $|u(x)| \leq x_2$*, where* $\mathbb{R}^2 = \{(x_1, x_2) : x_2 > 0\}$*. Show that* $u(x) = cx_2$ *for some constant* $c \in [-1, 1]$.

Problem 10 (6 point)**.** *Suppose that a smooth solution* $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *to the diffusion equation* $u_t = \Delta u + u^2$ *satisfies* $u(x,t) = u(x + e_i, t)$ *for each* $i \in \{1, \cdots, n\}$*. Show that* $u = 0$.

4. Maximum principle

Problem 11 (5 point)**.** *Let* $\Omega = B_1(0) \subset \mathbb{R}^2$*. Given a positive function* $f \in C^\infty(\overline{\Omega})$*, we suppose that a strictly convex smooth function* $u \in C^\infty(\overline{\Omega})$ *satisfies* $u = 0$ *on* $\partial B_1(0)$ *and*

$$\det \nabla^2 u(x) = f(x), \tag{18}$$

holds in $\overline{\Omega}$*, where* $\det \nabla^2 u = u_{11}u_{22} - u_{12}^2$*. Show that*

$$u(x) \geq -\tfrac{1}{2}(1 - |x|^2) \sup_{y \in \Omega} \sqrt{f(y)}, \tag{19}$$

holds for all $x \in \Omega$.

Problem 12 (5 point)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with smooth boundary. Suppose that a smooth solution* $u : \overline{Q}_T \to \mathbb{R}$ *(where* $Q_T = \Omega \times (0, T]$*) to the heat equation* $u_t = \Delta u$ *satisfies the boundary condition* $u = g$ *on* $\partial_p Q_T$ *for some* $g \in C^\infty(\overline{\Omega})$*. Show that if* g *satisfies* $g \geq 0$ *in* Ω *and* $g > 0$ *in* $\partial\Omega$*, then* $u(x,t) > 0$ *holds for* $t > 0$.